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FRACTURE MECHANICS OF ORTHOTROPIC LAMINATED PLATES—II. THE PLANE STRAIN CRACK PROBLEM FOR TWO BONDED ORTHOTROPIC LAYERS

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Abstract—In this paper the plane strain problem for two bonded orthotropic layers containing a crack perpendicular to the laminate surfaces is considered. After deriving the integral equation for the general problem, the solution is obtained for three main crack geometries, namely an embedded crack, a surface crack and a crack terminating at the interface under a remotely applied membrane load or bending moment. For the crack terminating at the interface the necessary asymptotic analysis is carried out, the characteristic equation to determine the power of stress singularity β is obtained, and the influence of the material parameters on β is examined. The main results given in the paper consist of the stress intensity factors obtained for various crack geometries, material combinations and loading conditions.

1. INTRODUCTION

The main objective of this series of papers is to study the fracture mechanics of orthotropic laminates under a combination of membrane and bending loads. Specifically, we aim to develop a technique for calculating the stress intensity factor along the front of a part-through crack in laminated plates in order to monitor the subcritical growth of surface flaws. Along with relative dimensions and loading conditions, an important factor to be studied will be the influence of the material orthotropy on the stress intensity factors. As indicated in Part I of this series of papers, the problem will be solved by extending the scope of the line spring model, which was originally developed for isotropic homogeneous plates, to laminated plates which consist of bonded dissimilar orthotropic layers. The procedure requires the formulation and solution of the corresponding through crack problem in orthotropic laminates and the development of the necessary compliance functions. To determine the compliance functions one needs the solution of the part-through crack problem in layered orthotropic materials under plane strain conditions. The through crack problem in orthotropic laminates by using various transverse shear deformation theories was considered in Part I. In this second part of the paper we will consider the plane strain problem in bonded orthotropic layers containing a surface crack. The synthesis of the two solutions through a line spring model for the purpose of treating the three-dimensional surface crack problem in orthotropic laminates will be considered in Part III of the paper.

The general plane strain problem for a layered medium containing a crack perpendicular to the laminate surfaces is described in Fig. 1. In elasticity problems involving layered dissimilar materials, the complexity of the solution increases rather rapidly with the number of dissimilar layers forming the laminate. In this problem we will, therefore, assume

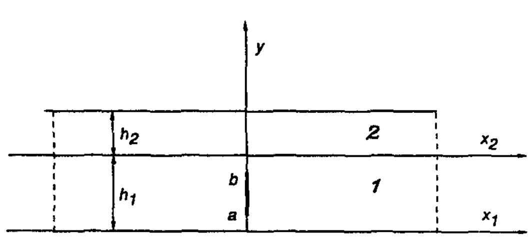

Fig. 1. Geometry and notation of the crack problem.

that the medium consists of only two dissimilar orthotropic layers. The formulation may also provide an approximation to a crack problem in a medium formed by an arbitrary number of orthotropic layers which are homogeneous on the two sides of the interface of primary interest. Usually, this dominant interface $y = y_d$ is assumed to be the interface which is nearest to the crack tip $y = b$ and for which $y_d > b$ (Fig. 1). Aside from providing the compliance functions needed for the line spring model, the plane problem described in Fig. 1, particularly in the limiting case of $a = 0$ and $b = h_1$, is a problem of great practical interest in its own right.

The plane elasticity problem for an orthotropic layer containing a crack perpendicular to the surface of the layer was considered by Delale and Erdogan (1977) and Kaya and Erdogan (1980) under various loading conditions. An unbounded layered medium which consists of two dissimilar periodically arranged bonded orthotropic layers containing periodic cracks perpendicular to the interfaces was studied by Delale and Erdogan (1979) who also gave some results for the crack terminating at and intersecting the interface.

2. THE FORMULATION OF THE PROBLEM

Consider the plane elasticity problem for two bonded orthotropic layers shown in Fig. 1. Assume that through an appropriate superposition the problem is reduced to a perturbation problem in which the crack surface tractions are the only nonzero external loads. We further assume that the applied loads are symmetric with respect to the $x = 0$ plane and hence the crack problem is one of mode I. The following equilibrium equations must be solved for each layer under the appropriate boundary and continuity conditions:

$$\beta_1 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \beta_3 \frac{\partial^2 v}{\partial x \, \partial y} = 0,$$
$$\frac{\partial^2 v}{\partial x^2} + \beta_2 \frac{\partial^2 v}{\partial y^2} + \beta_3 \frac{\partial^2 u}{\partial x \, \partial y} = 0, \tag{1}$$

where u and v are the x and y components of the displacement vector and the constants β_1, β_2 and β_3 are related to the engineering constants as follows:

$$\beta_1 = \frac{b_{11}}{b_{66}}, \quad \beta_2 = \frac{b_{22}}{b_{66}}, \quad \beta_3 = 1 + \frac{b_{12}}{b_{66}}, \tag{2}$$

$$(b_{ij}) = B = C^{-1}, \quad C = (c_{ij}), \quad (i, j) = (1,2), \quad b_{66} = G_{xy}, \tag{3}$$

$$c_{11} = \frac{1 - \nu_{xz} - \nu_{zx}}{E_x}, \quad c_{22} = \frac{1 - \nu_{yz} - \nu_{zy}}{E_y},$$
$$c_{12} = c_{21} = -\frac{\nu_{yx} + \nu_{zx}\nu_{yz}}{E_y} = -\frac{\nu_{xy} + \nu_{zy}\nu_{xz}}{E_x}, \tag{4a}$$

for plane strain, and

$$c_{11} = \frac{1}{E_x}, \quad c_{22} = \frac{1}{E_y}, \quad c_{12} = c_{21} = -\frac{\nu_{yx}}{E_y} = -\frac{\nu_{xy}}{E_x}, \tag{4b}$$

for generalized plane stress.

The stress–displacement relations are given by

$$\sigma_{xx} = b_{11}\frac{\partial u}{\partial x} + b_{12}\frac{\partial v}{\partial y},$$

$$\sigma_{yy} = b_{12}\frac{\partial u}{\partial x} + b_{22}\frac{\partial v}{\partial y},$$

$$\sigma_{xy} = b_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right). \tag{5}$$

Consider now the bonded orthotropic layers shown in Fig. 1. Due to symmetry, the problem will be considered for $0 < x < \infty$. Let (x_1, y_1) and (x_2, y_2) be the local axes for layers 1 and 2 as shown in the figure. For each layer the displacements may be expressed in terms of the following Fourier integrals:

$$\begin{aligned} u(x,y) = {} & \frac{2}{\pi}\int_0^\infty [K_1(\alpha)\sinh(s_1\alpha y) + K_2(\alpha)\cosh(s_1\alpha y) + K_3(\alpha)\sinh(s_2\alpha y) \\ & + K_4(\alpha)\cosh(s_2\alpha y)]\sin\alpha x\,\mathrm{d}\alpha \\ & + \frac{2}{\pi}\int_0^\infty [C_1(\gamma)\exp(-s_1\gamma x/\beta_5) + C_2(\gamma)\exp(-s_2\gamma x/\beta_5)]\cos\gamma y\,\mathrm{d}\gamma, \\ v(x,y) = {} & \frac{2}{\pi}\int_0^\infty [\beta_7 K_2(\alpha)\sinh(s_1\alpha y) + \beta_7 K_1(\alpha)\cosh(s_1 a y) \\ & + \beta_8 K_4(\alpha)\sinh(s_2\alpha y) + \beta_8 K_3(\alpha)\cosh(s_2\alpha y)]\cos\alpha x\,\mathrm{d}\alpha \\ & - \frac{2}{\pi}\int_0^\infty [\beta_9 C_1(\gamma)\exp(-s_1\gamma x/\beta_5) + \beta_{10}C_2(\gamma)\exp(-s_2\gamma x/\beta_5)]\sin\gamma y\,\mathrm{d}\gamma, \end{aligned} \tag{6}$$

where K_i and C_j ($i = 1, \ldots, 4$; $j = 1, 2$) are the unknown functions to be determined from the boundary conditions and s_1 and s_2 are the positive roots of the characteristic equation

$$s^4 + \beta_4 s^2 + \beta_5^2 = 0, \tag{7}$$

$$s_1 = +\sqrt{(-\beta_4+\beta_6)/2} = -s_3, \quad s_2 = +\sqrt{(-\beta_4-\beta_6)/2} = -s_4. \tag{8}$$

The coefficients $\beta_4, \ldots, \beta_9$ are given by:

$$\begin{aligned} & \beta_4 = \frac{\beta_3^2 - \beta_1\beta_2 - 1}{\beta_2}, \quad \beta_5^2 = \frac{\beta_1}{\beta_2}, \quad \beta_6 = \sqrt{\beta_4^2 - 4\beta_5^2}, \\ & \beta_7 = \frac{\beta_3 s_1}{1-\beta_2 s_1^2}, \quad \beta_8 = \frac{\beta_3 s_2}{1-\beta_2 s_2^2}, \\ & \beta_9 = -\frac{1}{\beta_3}\left[\frac{\beta_1 s_1}{\beta_5} - \frac{\beta_5}{s_1}\right], \quad \beta_{10} = -\frac{1}{\beta_3}\left[\frac{\beta_1 s_2}{\beta_5} - \frac{\beta_5}{s_2}\right]. \end{aligned} \tag{9}$$

We now define

$$C_h = -2 - \beta_4/\beta_5 \tag{10}$$

and observe that $\beta_5 > 0$. Thus, if $C_h > 0$ then $\beta_4 < 0$, $\beta_4^2 - 4\beta_5^2 > 0$, β_6 is real, $\beta_6 < -\beta_4$, and from (8) it follows that $s_1, \ldots, s_4$ are real. On the other hand, if $C_h < 0$ it may easily be shown that the roots s_i are always complex and have the form

$$s_1 = -s_3 = \omega_1 + i\omega_3, \quad s_2 = -s_4 = \omega_2 + i\omega_4, \quad \omega_1 > 0, \quad \omega_2 > 0. \tag{11}$$

The critical orthotropy constant C_h was defined by Chou (1962) and was shown to have the range $-4 < C_h < \infty$, $C_h = 0$ corresponding to the isotropic case. Note that C_h is equivalent to the constant κ defined by Krenk (1979) which may be expressed as

$$\kappa = -\frac{\beta_4}{2\beta_5}, \quad C_h = 2(\kappa - 1), \quad -1 < \kappa < \infty. \tag{12}$$

The orthotropic solids with $C_h > 0$ and $C_h < 0$ are generally known as materials type I and II, respectively. In the displacement expressions (6) it is assumed that the roots $s_1, \ldots, s_4$ are real. Similar expressions may be obtained for materials with complex roots.

3. THE INTEGRAL EQUATION

The solution as expressed by (6) has six unknown functions $K_i(\alpha)$ and $C_j(\alpha)$, $(i = 1, \ldots, 4; j = 1, 2)$ for each layer. Eleven of these 12 unknowns may be eliminated by using the homogeneous conditions which consist of four continuity conditions along the interface, four traction boundary conditions on the surfaces $y_1 = 0$ and $y_2 = h_2$, two symmetry conditions $\sigma_{2xy}(0, y_2) = 0$, $u_2(0, y_2) = 0$, $0 < y_2 < h_2$ in material II, and one symmetry condition $\sigma_{1xy}(0, y_1) = 0$, $0 < y_1 < h_1$ in material I (Fig. 1). The remaining unknown may be determined from the following mixed boundary conditions:

$$\sigma_{1xx}(0, y) = -p(y), \quad a < y < b,$$
$$u_1(0, y) = 0, \quad 0 < y < a, \quad b < y < h_1. \tag{13}$$

Defining a new unknown function by

$$\phi(y) = \frac{\partial}{\partial y} u_1(0, y), \quad 0 < y < h_1, \tag{14}$$

and replacing (13) by (14), it is seen that all 12 unknown functions and consequently all field quantities in the composite medium may be expressed in terms of $\phi(y)$. From (13b) and (14) it follows that

$$\int_a^b \phi(y)\,\mathrm{d}y = 0. \tag{15}$$

After determining the functions K_{ij} and C_{ik} $(i = 1, 2; j = 1, \ldots, 4; k = 1, 2)$ in terms of ϕ, by substituting from (6) and (5) into (13a), we obtain the integral equation to solve for $\phi(y)$.

Because of the large number of elastic constants and unknown functions involved, the process of deriving the integral equation is rather complicated and lengthy. However, the technique is very straightforward and is similar to that followed by Delale and Erdogan (1979). Thus, referring to Wu (1990) for details, the integral equation to determine $\phi(y)$ may be obtained as follows:

$$\frac{\pi}{\mu^*}\sigma_{1xx}(0, y) = \int_a^b \left[\frac{1}{t - y} + k(y, t)\right]\phi(t)\,\mathrm{d}t = -\frac{\pi}{\mu^*}p(y), \quad a < y < b, \tag{16}$$

where for the plane strain problem under consideration the material constant μ^*, which is a scaling parameter for the crack opening displacement, is given in terms of the coefficients of the stiffness matrix of layer 1 by

$$\mu^* = \left(\frac{b_{11}b_{12}}{2}\right)^{1/2}\left[\left(\frac{b_{22}}{b_{11}}\right)^{1/2} + \frac{2b_{12}+b_{66}}{2b_{11}}\right]^{1/2}. \tag{17}$$

Note that for an isotropic medium the corresponding constant is

$$\mu^* = \frac{4\mu}{1+\kappa}, \tag{18}$$

where μ is the shear modulus and $\kappa = 3-4\nu$ for plane strain and $\kappa = (3-\nu)/(1+\nu)$ for plane stress, ν being the Poisson's ratio. The kernel $k(y,t)$ which appears in (16) has the following form:

$$k(y,t) = \frac{1}{t+y} + \int_0^\infty K(y,t,\alpha)\,\mathrm{d}\alpha, \tag{19}$$

where the function $K(y,t,\alpha)$ can be found in Wu (1990). Referring to Fig. 1, if $a > 0$, then (16) must be solved under the single-valuedness condition (15). In the case of an edge crack $a = 0$, (15) is not valid and is replaced by the following boundedness condition:

$$|\phi(0)| < \infty. \tag{20}$$

In the case of a crack embedded in layer 1, that is, for $a > 0$, $b < h_1$, the kernel $k(y,t)$ in (16) is bounded in the closed interval $a \leqslant (y,t) \leqslant b$ and, consequently the solution of the singular integral equation has the form

$$\phi(t) = \frac{g(t)}{[(t-a)(b-t)]^{1/2}}, \quad a < t < b. \tag{21}$$

Thus, since (16) gives the expression for $\sigma_{1xx}(0,y)$ outside as well as within the crack, it may easily be shown that at the crack tips a and b, σ_{1xx} has a square root singularity. One can, therefore, define the mode I stress intensity factors and evaluate them in terms of $\phi(y)$ as follows:

$$\begin{aligned} k(a) &= \lim_{y\to a-}\sqrt{2(a-y)}\,\sigma_{1xx}(0,y) = \lim_{y\to a+}\mu^*\sqrt{2(y-a)}\,\phi(y), \\ k(b) &= \lim_{y\to b+}\sqrt{2(y-b)}\,\sigma_{1xx}(0,y) = -\lim_{y\to b-}\mu^*\sqrt{2(b-y)}\,\phi(y). \end{aligned} \tag{22}$$

For an edge crack, that is, for $a = 0$, $b < h_1$ assuming

$$\phi(t) = \frac{g(t)}{t^{\alpha}(b-t)^{\beta}}, \quad 0 < \mathrm{Re}\,(\alpha,\beta) < 1, \tag{23}$$

and performing the necessary asymptotic analysis (Muskhelishvili, 1953), it can be shown that $\cot \pi\beta = 0$ and there is no root in the strip $0 < \mathrm{Re}\,(\alpha) < 1$. Similarly, it can also be shown that at $t = 0$, $\phi(t)$ has no logarithmic singularity. Thus, the numerical solution of embedded and edge crack problems may be carried out by using the standard techniques [e.g. Erdogan (1978)].

Even though for the determination of compliance functions to be used in the line spring model the solution of the edge crack problem with $b < h_1$ is sufficient, an interesting and important limiting case of the plane elasticity problem described in Fig. 1 is the case in which $b = h_1$. In this case, despite the fact that the integral equation remains valid (with $b = h_1$ and $0 \leqslant a < h_1$), the kernel $k(y,t)$ becomes unbounded as both y and t approach the end point h_1 and contributes to the singular behavior of the function $\phi(y)$ at $y = h_1$. To determine the correct behavior of ϕ and, consequently, that of the stress state near the end point h_1 and to develop an effective method for solving the integral equation, it is necessary to separate the part of the kernel $k(y,t)$ which becomes singular as $(y,t) \to h_1$.

After performing the appropriate asymptotic analysis, the kernel $k(y,t)$ may be expressed as follows:

$$k(y,t) = k_s(y,t) + k_b(y,t), \tag{24}$$

where k_b is bounded in the closed interval $a \leqslant y \leqslant h_1$ and k_s is given by

$$k_s(y,t) = P_1 \frac{s_1 h_1 + (h_1 - t)\beta_{11}}{(s_1 h_1 + (h_1 - t)\beta_{11})^2 - (s_1 y)^2} + P_2 \frac{s_1 h_1 + (h_1 - t)\beta_{12}}{(s_1 h_1 + (h_1 - t)\beta_{12})^2 - (s_1 y)^2}$$
$$+ P_3 \frac{s_2 h_1 + (h_1 - t)\beta_{11}}{(s_2 h_1 + (h_1 - t)\beta_{11})^2 - (s_2 y)^2} + P_4 \frac{s_2 h_1 + (h_1 - t)\beta_{12}}{(s_2 h_1 + (h_1 - t)\beta_{12})^2 - (s_2 y)^2}, \tag{25}$$

$$\beta_{11} = \beta_5 / s_1, \quad \beta_{12} = \beta_5 / s_2. \tag{26}$$

In (25) P_i $(i = 1, \ldots, 4)$ are known material constants that come out of the asymptotic analysis [see Wu (1990) for details]. Note that for $t = h_1$, k_s behaves as $(h_1 - y)^{-1}$ and becomes unbounded for $y = h_1$. This end point behavior is quite typical of the generalized Cauchy kernels. Defining the unknown function $\phi(y)$ by

$$\phi(y) = \frac{g(y)}{(y-a)^{\alpha}(h_1 - y)^{\beta}}, \quad a < y < h_1, \quad (0 < \mathrm{Re}\,(\alpha, \beta) < 1), \tag{27}$$

substituting from (24), (25) and (27) into (16) and performing the asymptotic analysis, the characteristic equations giving α and β may be obtained as follows:

$$\cot \pi\alpha = 0, \tag{28}$$

$$2\cos(\pi\beta) - P_1 \frac{1}{\beta_{11}(s_1/\beta_{11})^{\beta}} - P_2 \frac{1}{\beta_{12}(s_1/\beta_{12})^{\beta}} - P_3 \frac{1}{\beta_{11}(s_2/\beta_{11})^{\beta}} - P_4 \frac{1}{\beta_{12}(s_2/\beta_{12})^{\beta}} = 0. \tag{29}$$

This is the same equation found by Delale and Erdogan (1979). The results for isotropic materials obtained, for example, by Cook and Erdogan (1972) may easily be duplicated by using (29). In this case, too, β turns out to be real and $\beta > \frac{1}{2}$ if layer 1 is stiffer than layer 2 and $\beta < \frac{1}{2}$ if layer 2 is the stiffer material.

As in similar previous studies, for the crack tip terminating at the interface, i.e. for $b = h_1$ in Fig. 1, the stress intensity factor must be defined in terms of the cleavage stress σ_{2xx} in layer 2, that is

$$k(h_1) = \lim_{y \to h_1 +} 2^{\beta}(y - h_1)^{\beta} \sigma_{2xx}(0, y). \tag{30}$$

The asymptotic expression of $\sigma_{2xx}(0,y)$ for small values of $(y - h_1)$ may be obtained by going back to the original formulation of the problem and deriving an expression for σ_{2xx} similar to (16), namely

$$\sigma_{2xx} = \frac{1}{\pi} \int_a^{h_1} k_2(y,t)\phi(t)\,\mathrm{d}t. \tag{31}$$

It can be shown that near $y = h_1$ k_2 has a singular part very similar to k_s given by (25) (Wu, 1990). Thus, by separating the singular part of kernel k_2, assuming

$$\phi(t) = \frac{g(t)}{(t-a)^{1/2}(h_1-t)^{\beta}}, \tag{32}$$

where β is known and is given by (29), and by performing the necessary asymptotic analysis, it may be shown that

$$k(h_1) = \mu_2^* \frac{g(h_1)}{h_1^{\beta}} \frac{1}{\sin \pi\beta} \left[\frac{P_1^*}{\beta_{11}} \left(\frac{\beta_{11}}{s_1^*} \right)^{\beta} + \frac{P_2^*}{\beta_{12}} \left(\frac{\beta_{12}}{s_1^*} \right)^{\beta} + \frac{P_3^*}{\beta_{11}} \left(\frac{\beta_{11}}{s_2^*} \right)^{\beta} + \frac{P_4^*}{\beta_{12}} \left(\frac{\beta_{12}}{s_2^*} \right)^{\beta} \right], \tag{33}$$

where s_1^* and s_2^* are the positive roots of the characteristic equation (7) for material 2, μ_2^* is obtained from (17) by using the stiffness coefficients of material 2, and P_j^* $(j = 1, \ldots, 4)$ and μ_2^* are known bimaterial constants (Wu, 1990).

4. RESULTS AND DISCUSSION

The problem described in Fig. 1 is solved for three typical crack geometries, namely the embedded crack $0 < a < b < h_1$, the edge crack $0 = a < b < h_1$ and the crack terminating at the interface, i.e. the broken layer $a = 0$, $b = h_1$. The crack surface traction $p(y)$ used in the perturbation problem as the input function is obtained from the solution of the uncracked laminate under the actual loading condition. For example, if the laminate is subjected to a remote membrane loading $N_x = N$, $p(y)$ is constant and is given by

$$p(y) = p_0 = \frac{N}{h_1 + \alpha_0 h_2}, \tag{34}$$

where

$$\alpha_0 = \frac{E_{2x}}{E_{1x}} \frac{1 - \nu_{1xz}\nu_{1zx}}{1 - \nu_{2xz}\nu_{2zx}} \tag{35}$$

for plane strain and

$$\alpha_0 = \frac{E_{2x}}{E_{1x}} \tag{36}$$

for plane stress corresponding to a "plate" and a "beam", respectively.

Similarly if the laminate is subjected to a uniform bending $M_{xx} = M$ away from the crack region, the stress in the uncracked medium may be expressed as follows:

$$\sigma_{1xx}(0, y) = p(y) = p_b \left(1 - \frac{y}{c_1} \right), \quad 0 < y < h_1, \tag{37}$$

where p_b is the stress at the surface $y = 0$ and is given by

$$p_b = \frac{3c_1 M}{c_1^3 + [(E_{2x}^*/E_{1x}^*) - 1](c_1 - h_1)^3 + (E_{2x}^*/E_{1x}^*)(h_1 + h_2 - c_1)^3}, \tag{38}$$

and the constant c_1 defines the location of the neutral axis

$$c_1 = \frac{E^*_{1x}h_1^2 + 2E^*_{2x}h_1h_2 + E^*_{2x}h_2^2}{2(E^*_{1x}h_1 + E^*_{2x}h_2)}, \tag{39}$$

$$E^*_{ix} = \frac{E_{ix}}{1 - \nu_{ixz}\nu_{izx}}, \quad i = 1, 2, \tag{40}$$

for plane strain, and

$$E^*_{ix} = E_{ix}, \quad (i = 1, 2) \tag{41}$$

for plane stress.

Note that the solution obtained by using the input functions (34) and (37) provide the basic results for a "constant" and a "linear" crack surface traction. Since the problem is linear, the solution for most other practical loading conditions may be obtained by properly scaling and superimposing these two basic results. For example, such loading conditions as uniform temperature changes and residual stresses may easily be expressed as the sum of a membrane and bending loadings.

The results given in this paper consist of the stress intensity factors defined by (22) and (30). However, by observing that μ^* defined in (17) is the scaling factor for the crack opening displacement $u_1(0, y)$, from the expression of the "strain energy release" for a crack growth db

$$dU = 2\int_b^{b+db} \tfrac{1}{2}\sigma_{1xx}(0, y)u_1(0, y - db)\,dy, \tag{42}$$

the "strain energy release rate" at the crack tip $y = b$ may also be obtained as

$$\mathscr{G} = \frac{dU}{db} = \frac{\pi k^2(b)}{2\mu^*}. \tag{43}$$

The elastic properties of the materials used in the examples are given in Table 1. Materials 1 and 2 are fiber-reinforced (graphite-epoxy) composites, 3 is steel and 4 is zirconia. Note that materials 1 and 2 are the same except for a 90° rotation about the y-axis (Fig. 1). Table 2

Table 1. Elastic constants of the materials used in examples (in units of GPA). Materials 1 and 2 are graphite-epoxy composites, 3 is steel and 4 is zirconia

Material 1	Material 2	Material 3	Material 4
$E_{1x} = 39.0$	$E_{2x} = 30.6$	$E = 200$	$E = 137.9$
$E_{1y} = 6.4$	$E_{2y} = 6.4$	$\nu = 0.26$	$\nu = 0.26$
$E_{1z} = 30.6$	$E_{2z} = 39.0$		
$G_{1xy} = 4.5$	$G_{2xy} = 4.5$		
$G_{1yz} = 4.5$	$G_{2yz} = 4.5$		
$G_{1xz} = 19.7$	$G_{2xz} = 19.7$		
$\nu_{1yx} = 0.275$	$\nu_{2xy} = 0.275$		
$\nu_{1zy} = 0.275$	$\nu_{2zy} = 0.275$		
$\nu_{1xz} = 0.447$	$\nu_{2xz} = 0.351$		

Table 2. Material combinations used in bonded layers (Fig. 1)

Material pair	Layer 1	Layer 2
A	2	1
B	1	2
C	4	3
D	3	4
I	3	3

Table 3. Stress intensity factors in a homogeneous isotropic layer of thickness h containing an edge crack of length b and subjected to membrane loading N or bending moment M. (Material pair I, $a = 0$, Fig. 1, $p_0 = N/h$, $p_b = 6M/h^2$)

$\frac{b}{(h/2)}$	$\frac{k_t}{p_0\sqrt{b}}$	$\frac{k_b}{p_b\sqrt{b}}$
0.001	1.1215	1.1215
0.1	1.1399	1.0708
0.2	1.1892	1.0472
0.3	1.2652	1.0432
0.4	1.3673	1.0553
0.5	1.4975	1.0826
0.6	1.6599	1.1241
0.7	1.8612	1.1826
0.8	2.1114	1.2606
0.9	2.4253	1.3630

Table 4. The influence of material constants on the power of stress singularity β for a crack tip terminating at the interface. The crack is in layer 1

Layer 1: Isotropic $\nu_1 = \nu_2 = 0.3$		Layer 1: Orthotropic $\mu_1^* = 1.35$ GPA		Layer 1: Orthotropic $\mu_1^* = 12.078$ GPA		Layer 1: Orthotropic $\mu_1^* = 61.6$ GPA	
μ_2^*/μ_1^*	β	μ_2^*/μ_1^*	β	μ_2^*/μ_1^*	β	μ_2^*/μ_1^*	β
0.001	0.963	$4.1 \cdot 10^{-6}$	0.998	0.045	0.835	$4.5 \cdot 10^{-6}$	0.998
0.01	0.915	$4.1 \cdot 10^{-5}$	0.995	0.119	0.755	$4.5 \cdot 10^{-5}$	0.995
0.045	0.826	$4.1 \cdot 10^{-4}$	0.986	0.375	0.650	$4.5 \cdot 10^{-4}$	0.984
0.1	0.246	$4.1 \cdot 10^{-3}$	0.954	0.659	0.564	0.0089	0.931
0.98	0.502	0.041	0.863	0.871	0.520	0.129	0.775
1.0	0.500	0.4075	0.664	1.000	0.500	0.196	0.725
1.02	0.498	0.998	0.520	3.642	0.346	0.999	0.528
10.0	0.333	1.00	0.500	5.10	0.313	1.000	0.500
22.22	0.301	5.90	0.286	13.66	0.242	17.86	0.279
44.44	0.300	7.79	0.277	91.06	0.193	89.27	0.254
100.0	0.294	41.0	0.157	273.2	0.186	892.7	0.248
1000.0	0.290	$4.1 \cdot 10^3$	0.121	546.6	0.184	$8.9 \cdot 10^3$	0.2477
		$4.1 \cdot 10^4$	0.117			$8.9 \cdot 10^4$	0.2477
		$4.1 \cdot 10^5$	0.117			$8.9 \cdot 10^5$	0.2477

Table 5. The effect of individual material constants on β. Layer 1 is isotropic with E, $\nu = 0.3$. Layer 2 has the same properties as layer 1 except for E_{2y} or G_{2xy}

E_{2y}/E	μ_2^*/μ_1^*	β	G_{2xy}/G_{1xy}	μ_2^*/μ_1^*	β
1	1	0.5	1	1	0.5
10	1.186	0.480	0.1	0.3667	0.603
100	12.88	0.459	0.01	0.1178	0.739

Table 6. The effect of individual material constants on β. Layer 1 is orthotropic; Material 1 is given in Table 1; Layer 2 has the same properties as layer 1 except for E_{2y} or G_{2xy}

E_{2x}/E_{1x}	μ_2^*/μ_1^*	β	G_{2xy}/G_{1xy}	μ_2^*/μ_1^*	β
0.5	0.6363	0.564	1	1	0.5
1	1	0.5	0.1	0.369	0.607
2	1.661	0.427	0.01	0.1192	0.744
5	5.1	0.283			

gives the material combinations used in the examples for the layered medium shown in Fig. 1. The four pairs A–D are selected in such a way that the crack may be located in any one of the four materials used. The fifth pair I corresponds to an isotropic layer of thickness $h = h_1 + h_2$ and is included to provide a benchmark. The stress intensity factors for the isotropic layer (material pair I) containing an edge crack of length b and subjected to membrane loading N or bending M are given in Table 3. These are essentially the results also obtained by Kaya and Erdogan (1980).

The calculated results for various material combinations, crack geometries and loading conditions are given in Figs 2–6 and Tables 4–6. In all the examples considered, the medium is assumed to be under plane strain conditions. For the bonded orthotropic layers (material pairs A and B) containing an embedded crack and subjected to a membrane loading N, the stress intensity factors under plane strain conditions are shown in Fig. 2. The normalized stress intensity factors shown in the figures are defined by

$$K_a = \frac{k(a)}{k_0}, \quad K_b = \frac{k(b)}{k_0}, \quad k_0 = p_0\sqrt{l}, \quad l = \frac{(b-a)}{2}, \tag{44}$$

where p_0 is the crack surface pressure given by (34). Note that in pair A the layer 2 and in pair B layer 1 is the stiffer medium and consequently, the stress intensity factors for pair A are seen to be smaller than for pair B. Also note that, for the same reason, the strength of the stress singularity β in pair A at $b = h_1$ ($\beta = 0.481$) is less than and in pair B ($\beta = 0.520$) greater than 1/2. From (16) it is seen that μ^* defined by (17) is the scaling factor for the crack opening displacement $u_1(0, y)$ [or for its derivative $\phi(y)$]. Thus, in the examples given in this study μ^* is used as a measure of the "stiffness" of the orthotropic medium. However, it should be strongly emphasized that, since the Fredholm kernel $k(y, t)$ in (16) is dependent on all the dimensionless material constants considered, μ_2^*/μ_1^* cannot be the only measure of the relative stiffness in bonded layers having a crack perpendicular to the interface.

Note that the definition of the conventional stress intensity factors is based on the square-root singularity. Thus, as the crack tip approaches the interface since the power of singularity changes from 1/2 to β, one would expect that as $b \to h_1$ the conventionally defined stress intensity factor would approach either zero or infinity depending on whether $\beta < 1/2$ or $\beta > 1/2$. Similarly, as $a \to 0$, due to the diminishing ligament size, $k(a)$ would be expected to become unbounded. These trends may indeed be observed in Fig. 2.

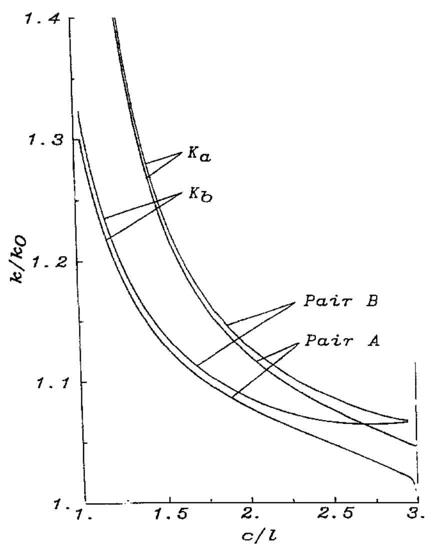

Fig. 2. The influence of the crack location on the stress intensity factors in two bonded orthotropic layers containing an embedded crack and subjected to uniform membrane loading away from the crack region, Fig. 1, $h_1 = h_2$, $k_0 = p_0\sqrt{l}$, $l = (b-a)/2 = h_1/4$, $c = (b+a)/2$; Material pair A: $\mu_2^*/\mu_1^* = 1.149$, $\beta = 0.481$; Material pair B: $\mu_2^*/\mu_1^* = 0.871$, $\beta = 0.520$.

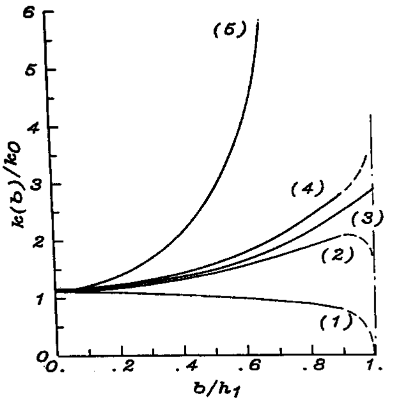

Fig. 3. The influence of the stiffness ratio and the crack length on the stress intensity factor in two bonded isotropic layers containing an edge crack and subjected to remote membrane loading. Figure 1, $a = 0$, $h_1 = h_2$, $k_0 = p_0\sqrt{b}$.

The influence of stiffness ratio E_2/E_1 and the crack length on the stress intensity factor in two bonded isotropic layers containing an edge crack and subjected to membrane loading N is shown in Fig. 3. The modulus ratio for the five material combinations used in these examples is as follows:

Material pair	(1)	(2)	(3)	(4)	(5)
E_2/E_1	40	1.45	1	0.69	0

In all five cases the Poisson's ratios are assumed to be $\nu_1 = \nu_2 = 0.3$. The material combinations (2), (3) and (4) correspond to, respectively, the material pairs C, I and D shown in Table 2. From the figure it may be observed that as $(b/h_1) \to 0$ the normalized stress intensity factor approaches 1.1215, the value for the semi-infinite isotropic plane having an edge crack. Also, as the crack tip approaches the interface, i.e. for $(b/h_1) \to 1$, the stress intensity factor tends to infinity for $E_2 < E_1$ and to zero for $E_2 > E_1$. Again, this is due to the change in the power of stress singularity at $b = h_1$.

Results similar to Fig. 3 involving orthotropic and isotropic layers are shown in Fig. 4. In these examples layer 1 is material 1 shown in Table 1 with $\mu^* = 12.078$ GPA. The stiffness ratios for five material combinations used in the examples shown in Fig. 4 are

Material pair	(1)	(2)	(3)	(4)	(5)
μ_2^*/μ_1^*	8.88	1	0.871	0.046	0

where μ_i^* $(i = 1, 2)$ for layers 1 and 2 are defined by (17). Here material pair (2) corresponds

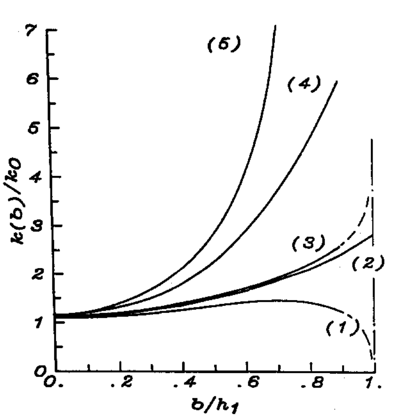

Fig. 4. The normalized stress intensity factor in two bonded orthotropic and isotropic layers containing an edge crack and subjected to membrane loading. Figure 1, $a = 0$, $h_1 = h_2$, $k_0 = p_0\sqrt{b}$, layer 1 is material 1 given in Table 1.

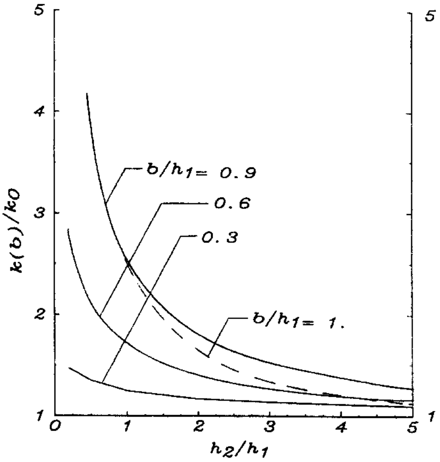

Fig. 5. The effect of thickness ratio on the stress intensity factor in two bonded orthotropic layers containing an edge crack and subjected to remote membrane loading. Figure 1, $a = 0$, $k_0 = p_0\sqrt{b}$, $k_h = p_0 h_1^\beta$, Material pair B, $\beta = 0.520$.

to a homogeneous orthotropic layer of thickness $h = h_1 + h_2$, (5) to a homogeneous layer of thickness h_1, and (3) to material pair B shown in Table 2. In material pair (4) and (1) the layer 2 is assumed to be an isotropic solid with $E_2 = 1$ GPA, $\nu_2 = 0.3$ and $E_2 = 200$ GPA, $\nu_2 = 0.26$, respectively. The stiffness coefficient μ^* is calculated from (17) or (18). The results given in Fig. 4 are qualitatively the same as that for Fig. 3 and, therefore, remarks made for Fig. 3 also apply to Fig. 4. In Fig. 4 as $(b/h_1) \to 0$ the half plane value for the stress intensity ratio 1.101 corresponding to the particular material (material 1, Table 1) is recovered.

Figure 5 shows the influence of the thickness ratio h_2/h_1 and the crack length on the stress intensity factor in two bonded orthotropic layers (the material pair B, Table 2) containing an edge crack and subjected to remote membrane loading. As expected, for fixed h_1 and b the stress intensity factor is a decreasing function of h_2. The figure also shows the stress intensity factor for the crack tip terminating at the interface, $b = h_1$, for which the power of stress singularity defined by (30) is $\beta = 0.520$.

The stress intensity factor in a two layer orthotropic laminate containing an edge crack and subjected to remote bending is shown in Fig. 6. The input function used in this example is given by (37).

Tables 4–6 show the effect of various material properties on the power of stress singularity β for a crack tip terminating at the interface. The crack is again in layer 1 and μ_1^* and μ_2^* are calculated from (17) for orthotropic and (18) for isotropic materials. In four different examples given in Table 4 it is assumed that layer 1 is fixed, in the first example as an isotropic material and in the remaining three as orthotropic solids. These three orthotropic materials are actual fiber-reinforced composites for which only μ_1^* is shown in the table (see, for example, Table 1, material 1 for the properties of the layer having

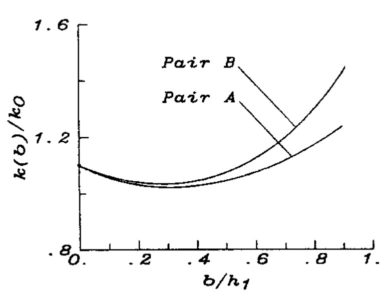

Fig. 6. Stress intensity factor in two bonded orthotropic layers containing an edge crack and subjected to uniform bending away from the crack region. Figure 1, $a = 0$, $h_1 = h_2$, $k_0 = p_b\sqrt{b}$.

$\mu_1^* = 12.078$ GPA). The first two columns corresponding to bonded isotropic layers simply duplicate the results given by Cook and Erdogan (1972). In the remainder of the table in some cases layer 2 is selected as an actual orthotropic material. But in most of the calculations layer 2 is assumed to be a hypothetical material with very wide range of stiffnesses selected for the purpose of demonstrating the variation in β. As $(\mu_2^*/\mu_1^*) \to 0$ the crack tends to become an "edge crack" terminating at a free surface and β approaches 1. On the other hand, as $(\mu_2^*/\mu_1^*) \to \infty$, β approaches a finite limit corresponding to the singularity at the apex of a 90° elastic wedge bonded to a rigid half plane (Hein and Erdogan, 1971).

Some sample results showing the effect of the individual material constants in layer 2 on the power of stress singularity β are shown in Tables 5 and 6. In Table 5 layer 1 is isotropic, in Table 6 it is a known orthotropic material (material 1, Table 1). In all cases layer 2 is identical to layer 1 with the exception of a single modulus, which is arbitrarily varied. The tables show the physically expected result, namely that an increase in any of the stiffnesses in layer 2 would cause a reduction in β and, conversely, decreased stiffness in layer 2 would result in an increase in β. The tables also show that the influence of the relative change in E_{2x} and G_{2xy} on β could be quite significant, whereas the effect of the variation in E_{2y} on β seems to be relatively small.

Coming back now to the main purpose of this paper, if the laminate containing an edge crack is subjected to remote membrane loading $N_x = N$ and bending $M_x = M$, by using the technique developed in the previous sections the stress intensity factor at the crack tip $y = b$ may be evaluated and expressed as follows:

$$k(b) = \sqrt{h}\left[\frac{N}{h} g_t(\xi) + \frac{M}{h^2/6} g_b(\xi)\right], \quad \xi = b/h, \tag{45}$$

where g_t and g_b are the dimensionless "shape functions". Assuming these functions to be in the form

$$g_t(\xi) = \sqrt{\xi}\sum_0^n c_{tk}\xi^k, \quad g_b(\xi) = \sqrt{\xi}\sum_0^n c_{bk}\xi^k, \tag{46}$$

the coefficients c_{tk} and c_{bk} may be obtained by an appropriate curve fitting. The functions g_t and g_b are then used to determine the compliance functions in the line spring model. More extensive results giving the plane strain stress intensity factors in various ceramic-coated metal substrates as well as in composites will be given in the third article.

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REFERENCES

Chou, Y. T. (1962). Interaction of parallel dislocations in a hexagonal crystal. *J. Appl. Phys.* **33**, 2747–2751.
Cook, T. S. and Erdogan, F. (1972). Stresses in bonded materials with a crack perpendicular to the interface. *Int. J. Engng Sci.* **10**, 677–697.
Delale, F. and Erdogan, F. (1977). The problem of internal and edge cracks in an orthotropic strip. *ASME J. Appl. Mech.* **44**, 237–242.
Delale, F. and Erdogan, F. (1979). Bonded orthotropic strips with cracks. *Int. J. Fract.* **15**, 343–364.
Erdogan, F. (1978). Mixed boundary-value problems in mechanics. In *Mechanics Today* (Edited by S. Nemat-Nasser), pp. 1–85. Pergamon Press, Oxford.
Hein, V. L. and Erdogan, F. (1971). Stress singularities in a two material wedge. *Int. J. Fract.* **7**, 317–330.
Kaya, A. C. and Erdogan, F. (1980). Stress intensity factors and COD in an orthotropic strip. *Int. J. Fract.* **16**, 171–190.
Krenk, S. (1979). On the elastic constants of plane orthotropic elasticity. *J. Compos. Mater.* **13**, 108–116.
Muskhelishvili, N. I. (1953). *Singular Integral Equations*. Noordhoff, Groningen, The Netherlands.
Wu, B. (1990). The surface and through crack problems in layered orthotropic plates. Ph.D. Dissertation, Lehigh University.